Material derivatives of higher dimension in geophysical fluid dynamics and in electrodynamics of fluids[1]

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Summary. The familiar operator in fluid dynamics D/Dt defines the material derivative for a fluid particle with dimension zero. In this paper we define and use "macroscopic" or multidimensional material derivatives D_1/Dt, D_2/Dt, and D_3/Dt. They are the material derivatives of infinitesimal properties of the fluid having dimensions, i. e. when particles build a line, a surface area, or a volume. Simple rules between the three operators are presented that avoid complicated calculations in fluid dynamics. For example, these operators are invariant with respect to solid rotations of coordinate systems. We rewrite a number of equations of fluid dynamics in terms of these operators and show that simple identities involving these operators already contain the structure of known vorticity theorems, especially those given by Hans ERTEL. One application deals with the circulation of eddy velocities in atmospheric turbulence, showing that this circulation may be an almost material invariant with time. Further possible applications (e.g., in electrodynamics and in radiation hydrodynamics) are also suggested.

Zusammenfassung. Die übliche materielle zeitliche Ableitung in der Hydrodynamik ist für eine Flüssigkeitspartikel definiert, die keine Dimension besitzt. In dieser Arbeit werden zusätzlich „makroskopische" materielle Ableitungen D_1/Dt, D_2/Dt, und D_3/Dt definiert und verwendet. Es handelt sich hierbei um materielle zeitliche Ableitungen von infinitesimalen Eigenschaften im Fluid, die Dimensionen besitzen, etwa solchen, die an materielle Linien, materielle Flächen oder an materielle Volumina gebunden sind. Zwischen den drei Operatoren bestehen einfache Beziehungen, welche helfen, viele komplizierte Rechnungen der Hydrodynamik zu umgehen. Besonders wichtig für das praktische Rechnen ist die Tatsache, dass diese Operatoren invariant gegenüber starren Rotationen von Koordinatensystemen sind. Eine Reihe wichtiger Gleichungen der Hydrodynamik lassen sich vorteilhaft unter Verwendung dieser Operatoren darstellen. Es wird gezeigt, dass einfache, mit Hilfe dieser Operatoren gebildete Identitäten bereits die Struktur bekannter Wirbelsätze, speziell diejenigen die Hans ERTEL angab, beschreiben. Eine Anwendung betrifft die Zirkulation der turbulenten Zusatzkomponenten der Strömung. Hier wird gezeigt, dass unter bestimmten Bedingungen diese Zirkulation materiell fast invariant sein kann. Weitere mögliche Anwendungen finden sich in der Dynamik und Energetik der elektromagnetischen Felder innerhalb eines Fluids sowie in der Strahlungshydrodynamik.

1. Introduction

Consider a material line element $d\mathbf{s}$, an element of a material surface area $d\mathbf{f}$, an element of a material volume $d\tau$, and an extensive local property $\mathbf{A}$ (a vector or dyadic) attached to each particle of these material elements of the fluid. "Macroscopic" properties of the fluid are: the line integral extended over a material line $S(t)$, the integral over a surface area (open surface) $F(t)$, and over a volume $V(t)$:

$$\int_{S_1}^{S_2} d\mathbf{s}\cdot\mathbf{A}, \quad \iint_{F(t)} d\mathbf{f}\cdot\mathbf{A}, \quad \iiint_{V(t)} d\tau\,\mathbf{A}.$$

These are "macroscopic" material structures of the fluid. They contain the *same particles all the time*. When the fluid moves, they depend on time. In this case, not only the variable $\mathbf{A}$ changes with time but the shapes of $d\mathbf{s}$, $d\mathbf{f}$, and $d\tau$ change too.

[1] In memory of Hans Ertel, the author´s teacher (1948-1951), his director (1951-1957), and his paternal friend thereafter.

An observer, moving with such a material structure, observes a rate of change with time D/Dt of it. At this stage of development, we remark that the derivative D/Dt applied to integrals emphasizes that we are following the motion of a "macroscopic" material structure. Therefore, in the following notation the operator D/Dt in front of integrals may not be identified with the common material derivative with time, i. e. with $D/Dt = \partial/\partial t + \mathbf{V}\cdot\nabla$, the material change with time of a "particle". It gets this meaning if we differentiate the integrands.

We know the following formulae at least from the end of the 19[th] century, see (MADELUNG, 1950), (FORTAK, 1956a, 1956b, 1960, 1967, 1993), (ARIS, 1962), and (KUNDU, 1990):

$$\frac{D}{Dt}\left\{\int_{s_1}^{s_2} d\mathbf{s}\cdot\mathbf{A}\right\} = \int_{s_1}^{s_2}\left\{d\mathbf{s}\cdot\frac{D\mathbf{A}}{Dt} + \frac{Dd\mathbf{s}}{Dt}\cdot\mathbf{A}\right\} = \int_{s_1}^{s_2} d\mathbf{s}\cdot\left\{\frac{D\mathbf{A}}{Dt} + (\nabla\mathbf{V})\cdot\mathbf{A}\right\} = \int_{s_1}^{s_2} d\mathbf{s}\cdot\frac{D_1\mathbf{A}}{\mathrm{Dt}}, \tag{1.1}$$

$$\frac{D}{Dt}\left\{\iint_{F(t)} d\mathbf{f}\cdot\mathbf{A}\right\} = \iint_{F(t)}\left\{d\mathbf{f}\cdot\frac{D\mathbf{A}}{Dt} + \frac{Dd\mathbf{f}}{Dt}\cdot\mathbf{A}\right\} = \iint_{F(t)} d\mathbf{f}\cdot\left\{\frac{D\mathbf{A}}{Dt} + (\nabla\cdot\mathbf{V})\mathbf{A} - (\mathbf{V}\nabla)\cdot\mathbf{A}\right\} = \iint_{F(t)} d\mathbf{f}\cdot\frac{D_2\mathbf{A}}{\mathrm{Dt}}, \tag{1.2}$$

$$\frac{D}{Dt}\left\{\iiint_{V(t)} d\tau\,\mathbf{A}\right\} = \iiint_{V(t)}\left\{d\tau\frac{D\mathbf{A}}{Dt} + \frac{Dd\tau}{Dt}\mathbf{A}\right\} = \iiint_{V(t)} d\tau\left\{\frac{D\mathbf{A}}{Dt} + (\nabla\cdot\mathbf{V})\mathbf{A}\right\} = \iiint_{V(t)} d\tau\frac{D_3\mathbf{A}}{\mathrm{Dt}}. \tag{1.3}$$

In contrast to the common zero dimensional material derivative, equations (1.1 to 1.3) define multi- dimensional material derivatives D_1/Dt, D_2/Dt, and D_3/Dt according to

$$\frac{1}{\Delta s}\frac{D}{Dt}\left\{\int_{s_1}^{s_2} d\mathbf{s}\cdot\mathbf{A}\right\}\to\frac{D_1\mathbf{A}}{Dt}\text{ as }\Delta s\to 0, \quad \frac{1}{\Delta F}\frac{D}{Dt}\left\{\iint_{\Delta F(t)} d\mathbf{f}\cdot\mathbf{A}\right\}\to\frac{D_2\mathbf{A}}{Dt}\text{ as }\Delta F\to 0,$$

$$\frac{1}{\Delta V}\frac{D}{Dt}\left\{\iiint_{V(t)} d\tau\,\mathbf{A}\right\}\to\frac{D_3\mathbf{A}}{Dt}\text{ as }\Delta V\to 0.$$

ΔS denotes the length of the material line $S(t)$, ΔF the area of the material surface $F(t)$, and ΔV the volume of the material volume $V(t)$. A proof of equations (1.1) to (1.3.) is given in Appendix 1.

These equations show how in material space of fluid dynamics the material derivative of a material integral may be interchanged with the integral itself. They correspond to what can be called "commutator" relationships. This is a very valuable property of the three introduced operators.

There are three important and extremely advantageous relationships, that can be verified quite easily:

$$\nabla\left(\frac{D\mathbf{A}}{Dt}\right) = \frac{D_1}{Dt}(\nabla\mathbf{A}), \qquad \nabla\times\left(\frac{D_1\mathbf{A}}{Dt}\right) = \frac{D_2}{Dt}(\nabla\times\mathbf{A}), \qquad \nabla\cdot\left(\frac{D_2\mathbf{A}}{Dt}\right) = \frac{D_3}{Dt}(\nabla\cdot\mathbf{A}). \tag{1.4}$$

A proof of (1.4) will be given in Appendix 2.

The equation of continuity allows a transformation of the operator D_3/Dt into D/Dt and vice versa. To start with, we write ($\upsilon = 1/\rho$ is specific volume):

$$\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{V} = \frac{D_3\rho}{Dt} = 0, \qquad \frac{D\upsilon}{Dt} - \upsilon\nabla\cdot\mathbf{V} = 0,.$$

Consequently,

$$\upsilon\frac{D_3\mathbf{A}}{Dt} = \frac{D(\upsilon\mathbf{A})}{Dt}, \tag{1.5}$$

$$\rho\frac{D\mathbf{A}}{Dt}=\frac{D_3(\rho\mathbf{A})}{Dt}=\frac{D(\rho\mathbf{A})}{Dt}+(\nabla\cdot\mathbf{V})(\rho\mathbf{A})=\frac{\partial}{\partial t}(\rho\mathbf{A})+\nabla\cdot(\mathbf{V}\rho\mathbf{A}). \tag{1.6}$$

These formulae help to avoid almost completely tedious calculations in fluid dynamics. They serve to formulate integral versions of hydrodynamic principles, and allow deriving many types of conservation principles. This is another useful property of the three introduced operators. By including the familiar total time derivative into the present formalism, one could rename D/Dt by D_0/Dt (not done here, but see (FORTAK, 1956). Equations (1.1 to 1.6) are the foundation for all calculations that will follow.

2. Transformation of the "macroscopic" material derivatives

As an important first example for the advantageous properties of the three operators, we mention their *invariance with respect to solid rotation* of coordinate systems.

Let be χ a scalar, and as before, $\mathbf{A}$ a vector or tensor. The index "a" indicates that a property is taken in an inertial ("absolute") system. A missing index "a" indicates the system fixed at the earth (relative system). The vector of the angular velocity of the earth rotation is $\boldsymbol{\omega}$. From the theorem of Coriolis the following equations should be used for transformation:

$$\frac{D_a\chi}{Dt}=\frac{D\chi}{Dt},\qquad \frac{D_a\mathbf{A}}{Dt}=\frac{D\mathbf{A}}{Dt}+\boldsymbol{\omega}\times\mathbf{A},\qquad \frac{D_a\mathbf{R}}{Dt}=\frac{D\mathbf{R}}{Dt}+\boldsymbol{\omega}\times\mathbf{R},\qquad \mathbf{V}_a=\mathbf{V}+\mathbf{V}_f. \tag{2.1}$$

Here **R** is the position vector, $\mathbf{V}_a$ the velocity vector in the inertial system, $\mathbf{V}$ the one in the relative system, and $\mathbf{V}_f$ is the velocity vector of solid rotation.

The kinematics of $\mathbf{V}_f$ is given as follows. The position vector is **R**, the unit dyadic is **I**, with: $\nabla\cdot\mathbf{R}=3,\ \nabla\times\mathbf{R}=0,\ \nabla\mathbf{R}=\mathbf{I}$. Therefore:

$$\begin{aligned}&\mathbf{V}_f=\boldsymbol{\omega}\times\mathbf{R},\quad \nabla\mathbf{V}_f=-\mathbf{I}\times\boldsymbol{\omega}=-\boldsymbol{\omega}\times\mathbf{I}\quad \mathbf{V}_f\nabla=\boldsymbol{\omega}\times\mathbf{I}=\mathbf{I}\times\boldsymbol{\omega}\\&\nabla\mathbf{V}_f+\mathbf{V}_f\nabla=0,\quad \nabla\mathbf{V}_f-\mathbf{V}_f\nabla=-2\boldsymbol{\omega}\times\mathbf{I},\quad \nabla\cdot\mathbf{V}_f=0,\quad \nabla\times\mathbf{V}_f=2\boldsymbol{\omega}\end{aligned}. \tag{2.2}$$

Important is that the dyadic $\nabla\mathbf{V}_f$ is antisymmetric. The transformation in (2.1) now writes as:

$$\frac{D_a\mathbf{A}}{Dt}=\frac{D\mathbf{A}}{Dt}+\left(\mathbf{V}_f\nabla\right)\cdot\mathbf{A}. \tag{2.3}$$

Applying equations (2.2), the transformation of the "macroscopic" material derivatives now is given by:

$$\frac{D_{1,a}\mathbf{A}}{Dt}=\frac{D_a\mathbf{A}}{Dt}+(\nabla\mathbf{V}_a)\cdot\mathbf{A}=\frac{D\mathbf{A}}{Dt}+\left(\mathbf{V}_f\nabla\right)\cdot\mathbf{A}+(\nabla\mathbf{V})\cdot\mathbf{A}+\left(\nabla\mathbf{V}_f\right)\cdot\mathbf{A}=\frac{D_1\mathbf{A}}{Dt}$$

$$\frac{D_{2,a}\mathbf{A}}{Dt}=\frac{D_a\mathbf{A}}{Dt}+(\nabla\cdot\mathbf{V}_a)\mathbf{A}-(\mathbf{V}_a\nabla)\cdot\mathbf{A}=\frac{D\mathbf{A}}{Dt}+\left(\mathbf{V}_f\nabla\right)\cdot\mathbf{A}+(\nabla\cdot\mathbf{V})\mathbf{A}-(\mathbf{V}\nabla)\cdot\mathbf{A}-\left(\mathbf{V}_f\nabla\right)\cdot\mathbf{A}=\frac{D_2\mathbf{A}}{Dt}$$

$$\frac{D_{3,a}\mathbf{A}}{Dt}=\frac{D_a\mathbf{A}}{Dt}+(\nabla\cdot\mathbf{V}_a)\mathbf{A}=\frac{D\mathbf{A}}{Dt}+(\nabla\cdot\mathbf{V})\mathbf{A}+\left(\mathbf{V}_f\nabla\right)\cdot\mathbf{A}=\frac{D_3\mathbf{A}}{Dt}+\left(\mathbf{V}_f\nabla\right)\cdot\mathbf{A}=\frac{D_3\mathbf{A}}{Dt}+\boldsymbol{\omega}\times\mathbf{A}$$

$$\frac{D_{3,a}\chi}{Dt}=\frac{D_3\chi}{Dt}$$

This important result states that the "macroscopic" material derivatives are invariant with regard to solid rotations of the system of coordinates. It should be noted that invariance of this kind holds in the third equation only, if the vector or tensor is replaced by a scalar. It will be seen that invariance of the three operators with regard to solid rotations of the coordinate system simplifies the derivation of vorticity equations and that of other equations in geophysical fluid dynamics enormously.

3. Fundamental equations of fluid dynamics rewritten

In an absolute frame of reference the Eulerian equations of motion are given as:

$$\frac{D_{\mathbf{a}}\mathbf{V}_{\mathbf{a}}}{Dt} = -\nabla\phi^{(A)} - \upsilon\nabla p + \upsilon\nabla\cdot\mathbf{F} = -\nabla\left(\phi^{(A)} + h\right) + T\nabla s + \upsilon\nabla\cdot\mathbf{F}\,.$$

The symbols have the usual meaning ($\phi^{(A)}$ stands for the attraction potential $\boldsymbol{F}$ for the friction tensor, h is specific enthalpy, s is specific entropy). Consistently with the formulae above we write:

$$\frac{D_{1,\mathbf{a}}\mathbf{V}_{\mathbf{a}}}{Dt} = \frac{D_1\mathbf{V}_{\mathbf{a}}}{Dt} = \nabla\left(\frac{\mathbf{V}_{\mathbf{a}}^2}{2} - \phi^{(A)}\right) - \upsilon\nabla p + \upsilon\nabla\cdot\mathbf{F} = \nabla\left\{\frac{\mathbf{V}_{\mathbf{a}}^2}{2} - \left(\phi^{(A)} + h\right)\right\} + T\nabla s + \upsilon\nabla\cdot\mathbf{F} = \mathbf{C}(\mathbf{V}_{\mathbf{a}})\,. \quad (3.1)$$

This is a first physical equation in D_1/Dt. Integration over a material line, with (1.1), the corresponding circulation theorem is obtained.

Applying the operation $\nabla\times$ on (3.1), equation (1.4) immediately leads to the equation for absolute vorticity $\mathbf{Z}_{\mathbf{a}} = \nabla\times\mathbf{V}_{\mathbf{a}} = \nabla\times\mathbf{V} + 2\boldsymbol{\omega}$

$$\frac{D_2\mathbf{Z}_{\mathbf{a}}}{Dt} = \nabla\times\mathbf{C}(\mathbf{V}_{\mathbf{a}})\,. \quad (3.2)$$

In addition, two physical equations in D_1/Dt are obtained from (3.1) as follows (e.g. FORTAK, 1993). The first equation is appropriate to *barotropic* flow (superscript *bt*), the second one to *adiabatic* flow (superscript *bk*). We begin with

$$\mathbf{C}^{bt}(\mathbf{V}_{\mathbf{a}}) = \nabla\left\{\frac{\mathbf{V}_{\mathbf{a}}^2}{2} - \left(\phi^{(A)} + P\right)\right\} - \upsilon\nabla p + \nabla P + \upsilon\nabla\cdot\mathbf{F}\,, \qquad P = \int^{p}\upsilon\,dp\,.$$

We introduce LAGRANGE´s function $L_{\mathbf{a}}^{bt}$ and HAMILTON´s action function $W_{\mathbf{a}}^{bt}$ according to:

$$L_{\mathbf{a}}^{bt} = \frac{\mathbf{V}_{\mathbf{a}}^2}{2} - \left(\phi^{(A)} + P\right) = \frac{DW_{\mathbf{a}}^{bt}}{Dt}\,, \quad W_{\mathbf{a}}^{bt} = \int_{t_0}^{t} dt\, L_{\mathbf{a}}^{bt}\,, \quad W_{\mathbf{a}}^{bt}\Big|_{t_0} = 0\,, \quad \nabla L_{\mathbf{a}}^{bt} = \frac{D_1}{Dt}\left(\nabla W_{\mathbf{a}}^{bt}\right). \quad (3.3)$$

Then, equation (3.1), the Eulerian equation of motion, in full generality writes as:

$$\frac{D_1}{Dt}\left(\mathbf{V}_{\mathbf{a}} - \nabla W_{\mathbf{a}}^{bt}\right) = -\upsilon\nabla p + \nabla P + \upsilon\nabla\cdot\mathbf{F}\,. \quad (3.4)$$

The associated vorticity equation is obtained at once by applying (1.4):

$$\frac{D_2\mathbf{Z}_{\mathbf{a}}}{Dt} = \nabla\times\left(-\upsilon\nabla p + \upsilon\nabla\cdot\mathbf{F}\right) = \mathbf{N}(p,\upsilon) + \nabla\times\left(\upsilon\nabla\cdot\mathbf{F}\right). \quad (3.4a)$$

Here $\mathbf{N}(p,\upsilon)$ is the *solenoid vector*. If the fluid is ideal and *barotropic*, we have conservation principles:

$$\frac{D_1}{Dt}\left(\mathbf{V}_{\mathbf{a}} - \nabla W_{\mathbf{a}}^{bt}\right) = 0\,, \qquad \frac{D_2\mathbf{Z}_{\mathbf{a}}}{Dt} = 0\,. \quad (3.5, 3.5a)$$

The second possibility (superscript *bk for baroclinic*) in writing the right-hand side of the equation of motion is given by

$$\mathbf{C}^{bk}(\mathbf{V}_{\mathbf{a}}) = \nabla\left\{\frac{\mathbf{V}_{\mathbf{a}}^2}{2} - \left(\phi^{(A)} + h\right)\right\} + T\nabla s + \upsilon\nabla\cdot\mathbf{F}\,. \quad (3.6)$$

First, we replace temperature by the function β (HELMHOLTZ, 1895). This useful function is a Lagrange multiplier of a general variational principle (SERRIN, 1959):

$$T = \frac{D\beta}{Dt}, \quad \beta = \int_{t_0}^{t} dt\, T, \quad \beta|_{t_0} = 0. \tag{3.7}$$

We further introduce LAGRANGE´s function $L_{\mathbf{a}}^{bk}$ and HAMILTON´s action function $W_{\mathbf{a}}^{bk}$:

$$L_{\mathbf{a}}^{bk} = \frac{\mathbf{V}_{\mathbf{a}}^2}{2} - \left(\phi^{(A)} + h\right) = \frac{DW_{\mathbf{a}}^{bk}}{Dt}, \quad W_{\mathbf{a}}^{bk} = \int_{t_0}^{t} dt\, L_{\mathbf{a}}^{bk}, \quad W_{\mathbf{a}}^{bk}\Big|_{t_0} = 0, \quad \nabla L_{\mathbf{a}}^{bk} = \frac{D_1}{Dt}\left(\nabla W_{\mathbf{a}}^{bk}\right). \tag{3.8}$$

Observing that

$$\frac{D\beta}{Dt}\nabla s = \frac{D_1}{Dt}(\beta\nabla s) - \beta\nabla\frac{Ds}{Dt},$$

we write equation (3.1) as

$$\frac{D_1}{Dt}\left(\mathbf{V}_{\mathbf{a}} - \beta\nabla s - \nabla W_{\mathbf{a}}^{bk}\right) = -\beta\nabla\frac{Ds}{Dt} + \upsilon\nabla\cdot\mathbf{F}. \tag{3.9}$$

The associated vorticity equation again is obtained by applying (1.4):

$$\frac{D_2}{Dt}\left(\mathbf{Z}_{\mathbf{a}} - \nabla\beta\times\nabla s\right) = \nabla\times\left\{-\beta\nabla\frac{Ds}{Dt} + \upsilon\nabla\cdot\mathbf{F}\right\}. \tag{3.9a}$$

If the fluid is ideal (i.e., without friction) and the motion is adiabatic, we have conservation principles:

$$\frac{D_1}{Dt}\left(\mathbf{V}_{\mathbf{a}} - \beta\nabla s - \nabla W_{\mathbf{a}}^{bk}\right) = 0, \qquad \frac{D_2}{Dt}\left(\mathbf{Z}_{\mathbf{a}} - \nabla\beta\times\nabla s\right) = 0. \tag{3.10, 3.10a}$$

Equations (3.4), (3.9) are related to transformations of WEBER (1868)) and of CLEBSCH (1857, 1859) (see e.g., SERRIN, 1959). We will shortly trace this purely theoretical subject in Appendix 1 because this was part of ERTEL´s scientific interests (ERTEL, **124**)[2].

We continue and summarize: Fluid dynamics is governed basically by two versions of the equations of motion. In abbreviated version, this is

$$\frac{D_1\mathbf{b}}{Dt} = \mathbf{C}(\mathbf{b}), \qquad \frac{D_2}{Dt}(\nabla\times\mathbf{b}) = \frac{D_2\mathbf{a}}{Dt} = \nabla\times\mathbf{C}(\mathbf{b}). \tag{3.11}$$

Here, we can use

$$\mathbf{b} = \mathbf{V}_{\mathbf{a}} - \nabla W_{\mathbf{a}}^{bt}, \quad \mathbf{a} = \mathbf{Z}_{\mathbf{a}}, \quad \mathbf{C}^{bt}(\mathbf{b}) = -\upsilon\nabla p + \nabla P + \upsilon\nabla\cdot\mathbf{F}, \tag{3.12}$$

$$\mathbf{b} = \mathbf{V}_{\mathbf{a}} - \beta\nabla s - \nabla W_{\mathbf{a}}^{bk}, \quad \mathbf{a} = \mathbf{Z}_{\mathbf{a}} - \nabla\beta\times\nabla s, \quad \mathbf{C}^{bk}(\mathbf{b}) = -\beta\nabla\frac{Ds}{Dt} + \upsilon\nabla\cdot\mathbf{F}. \tag{3.13}$$

For ideal fluids, $\mathbf{C}^{bt}(\mathbf{b}) = 0$ if the flow is *barotropic*, $\mathbf{C}^{bk}(\mathbf{b}) = 0$ if the flow is *adiabatic*.

There seems not to be a second D_2/Dt-equation, in addition to the vorticity equations, does not seem to exist in fluid dynamics. But there are D_3/Dt-equations. The equation of continuity is one example. The entropy equation is another one. Applying (1.6) from:

$$\rho T\frac{Ds}{Dt} = -\nabla\cdot\mathbf{J}_q + \nabla\mathbf{V}\cdot\cdot\mathbf{F},$$

we have

[2] The bold numbers attached to Ertel´s papers refer to the Dr. Gertrud Kobe list of Ertel´s publications. In this volume.

$$\frac{D_3\rho s}{Dt}+\nabla\cdot\left(\frac{\mathbf{J_q}}{T}\right)=\frac{1}{T}\left(-\frac{\mathbf{J_q}}{T}\cdot\nabla T+\mathbf{F}\cdot\cdot\nabla\mathbf{V}\right)=\rho\sigma\,. \tag{3.14}$$

Here, $\mathbf{J_q}$ is the molecular heat flux vector and $\mathbf{J_q}/T$ the corresponding entropy flux vector (radiation forcing is neglected here which does not touch the present argument). The right-hand side of (3.14) is entropy production $\rho\sigma$ (= 0 for reversible processes, > 0 for irreversible ones). The equation for total energy e (kinetic, potential, and internal energies) is written as

$$\frac{D_3\rho e}{Dt}+\nabla\cdot\mathbf{J_q}=\nabla\cdot\left[\left(-p\mathbf{I}+\mathbf{F}\right)\cdot\mathbf{V}\right]. \tag{3.15}$$

4. Identities and general vorticity theorems

The lasting value of ERTEL´s work is that he derived a number of general vorticity theorems for the first time. Ertel´s theorems became re-derived many times; sometimes applying complicated mathematics (see SCHRÖDER et al., 1993). In this paper, we refer back to some of ERTEL´s papers in this field. Most of them are available in English translation enclosed in this volume of the journal (SCHUBERT et al., 2004).

We will show that there is one common origin for (almost) all of them, and, additionally, that in fact they are based on one identity that is written in terms of the introduced "macroscopic" operators.

If we use a vector $\mathbf{a}$ and an arbitrary dyadic or tensor $\mathbf{A}$ **(**covariant on his left side in connection with D_1/Dt, and cotravariant in connection with D_2/Dt**)**, we can define useful *identities*. Let be $\mathbf{a\,A}$ a dyadic product, then:

$$\left\{\frac{D(\mathbf{a\,A})}{Dt}+(\nabla\cdot\mathbf{V})(\mathbf{a\,A})-(\mathbf{V}\nabla)\cdot(\mathbf{a\,A})\right\}-\mathbf{a}\frac{D\mathbf{A}}{Dt}=\left\{\frac{D\mathbf{a}}{Dt}+(\nabla\cdot\mathbf{V})\mathbf{a}-\mathbf{a}\cdot(\nabla\mathbf{V})\right\}\mathbf{A}$$

$$\frac{D_2(\mathbf{aA})}{Dt}-\mathbf{a}\frac{D\mathbf{A}}{Dt}=\frac{D_2\mathbf{a}}{Dt}\mathbf{A} \tag{4.1}$$

If we use as physical input equations (3.12), and (3.13), we get two vorticity theorems for the dyadic $\mathbf{aA}$.

If the dyadic product $\mathbf{a\,A}$ is replaced by $\mathbf{a}\chi$ with a scalar function χ, it follows that:

$$\frac{D_2(\chi\mathbf{Z_a})}{Dt}-\mathbf{Z_a}\frac{D\chi}{Dt}=\chi\nabla\times\mathbf{C}^{bt}(\mathbf{b})\ \ (=0:\ \text{ideal and barotropic})$$
$$\frac{D_2}{Dt}\left\{\chi(\mathbf{Z_a}-\nabla\beta\times\nabla s)\right\}-(\mathbf{Z_a}-\nabla\beta\times\nabla s)\frac{D\chi}{Dt}=\chi\nabla\times\mathbf{C}^{bk}(\mathbf{b})\ \ (=0:\ \text{ideal and adiabatic}) \tag{4.2}$$

are two generalizations of a result in (ERTEL, **96**). Integration over open or closed surfaces, and applying (1.2) leads to circulation principles depending on the special circumstances involved. These are simple for cases in which χ is materially invariant and if the flow of an ideal fluid is barotropic or adiabatic.

Further important identity is given by

$$\left\{\frac{D(\mathbf{a}\cdot\mathbf{A})}{Dt}+(\nabla\cdot\mathbf{V})(\mathbf{a}\cdot\mathbf{A})\right\}-\mathbf{a}\cdot\left\{\frac{D\mathbf{A}}{Dt}+(\nabla\mathbf{V})\cdot\mathbf{A}\right\}=\left\{\frac{D\mathbf{a}}{Dt}+(\nabla\cdot\mathbf{V})\mathbf{a}-\mathbf{a}\cdot(\nabla\mathbf{V})\right\}\cdot\mathbf{A}\,,$$

$$\frac{D_3(\mathbf{a}\cdot\mathbf{A})}{Dt}-\mathbf{a}\cdot\frac{D_1\mathbf{A}}{Dt}=\frac{D_2\mathbf{a}}{Dt}\cdot\mathbf{A}\,. \tag{4.3}$$

$$\frac{D(\upsilon\mathbf{a}\cdot\mathbf{A})}{Dt}-\upsilon\mathbf{a}\cdot\frac{D_1\mathbf{A}}{Dt}=\upsilon\frac{D_2\mathbf{a}}{Dt}\cdot\mathbf{A} \tag{4.4}$$

Introducing the physical input (3.12), (3.13), we obtain:

$$\frac{D_3(\mathbf{Z_a}\cdot\mathbf{A})}{Dt}-\mathbf{Z_a}\cdot\frac{D_1\mathbf{A}}{Dt}=\left\{\nabla\times\mathbf{C}^{bt}(\mathbf{b})\right\}\cdot\mathbf{A}$$
$$\frac{D(\upsilon\mathbf{Z_a}\cdot\mathbf{A})}{Dt}-\upsilon\mathbf{Z_a}\cdot\frac{D_1\mathbf{A}}{Dt}=\upsilon\left\{\nabla\times\mathbf{C}^{bt}(\mathbf{b})\right\}\cdot\mathbf{A} \tag{4.5}$$

$$\frac{D_3}{Dt}\left\{(\mathbf{Z_a}-\nabla\beta\times\nabla s)\cdot\mathbf{A}\right\}-(\mathbf{Z_a}-\nabla\beta\times\nabla s)\cdot\frac{D_1\mathbf{A}}{Dt}=\left\{\nabla\times\mathbf{C}^{bk}(\mathbf{b})\right\}\cdot\mathbf{A}$$
$$\frac{D}{Dt}\left\{\upsilon(\mathbf{Z_a}-\nabla\beta\times\nabla s)\cdot\mathbf{A}\right\}-\upsilon(\mathbf{Z_a}-\nabla\beta\times\nabla s)\cdot\frac{D_1\mathbf{A}}{Dt}=\upsilon\left\{\nabla\times\mathbf{C}^{bk}(\mathbf{b})\right\}\cdot\mathbf{A} \tag{4.6}$$

According to (3.5), (3.5a), and (3.10), (3.10a) for barotropic or adiabatic flows of ideal fluids, we obtain equations for *generalized helicity* if we choose $\mathbf{A}=\mathbf{V_a}-\nabla W^{bt}$ and $\mathbf{A}=\mathbf{V_a}-\beta\nabla s-\nabla W^{bk}$

$$\frac{D_3}{Dt}\left\{\mathbf{Z_a}\cdot\left(\mathbf{V_a}-\nabla W^{bt}\right)\right\}=0,$$
$$\frac{D}{Dt}\left\{\upsilon\mathbf{Z_a}\cdot\left(\mathbf{V_a}-\nabla W^{bt}\right)\right\}=0$$
$$\frac{D_3}{Dt}\left\{(\mathbf{Z_a}-\nabla\beta\times\nabla s)\cdot\left(\mathbf{V_a}-\beta\nabla s-\nabla W^{bk}\right)\right\}=0$$
$$\frac{D}{Dt}\left\{\upsilon(\mathbf{Z_a}-\nabla\beta\times\nabla s)\cdot\left(\mathbf{V_a}-\beta\nabla s-\nabla W^{bk}\right)\right\}=0 . \tag{4.7}$$

If specifically, we introduce $\mathbf{A}=\nabla\mathbf{B}$ into (4.3), together with (1.4), we obtain the identity.

$$\frac{D_3(\mathbf{a}\cdot\nabla\mathbf{B})}{Dt}-\mathbf{a}\cdot\nabla\left(\frac{D\mathbf{B}}{Dt}\right)=\frac{D_2\mathbf{a}}{Dt}\cdot\nabla\mathbf{B}. \tag{4.8}$$

Again, we introduce the physical input (3.12), (3.13)

$$\frac{D_3(\mathbf{Z_a}\cdot\nabla\mathbf{B})}{Dt}-\mathbf{Z_a}\cdot\nabla\left(\frac{D\mathbf{B}}{Dt}\right)=\left(\nabla\times\mathbf{C}^{bt}(\mathbf{b})\right)\cdot\nabla\mathbf{B}$$
$$\frac{D(\upsilon\mathbf{Z_a}\cdot\nabla\mathbf{B})}{Dt}-\upsilon\mathbf{Z_a}\cdot\nabla\left(\frac{D\mathbf{B}}{Dt}\right)=\upsilon\left(\nabla\times\mathbf{C}^{bt}(\mathbf{b})\right)\cdot\nabla\mathbf{B} , \tag{4.9}$$

$$\frac{D_3}{Dt}\left\{(\mathbf{Z_a}-\nabla\beta\times\nabla s)\cdot\nabla\mathbf{B}\right\}-(\mathbf{Z_a}-\nabla\beta\times\nabla s)\cdot\nabla\left(\frac{D\mathbf{B}}{Dt}\right)=\left(\nabla\times\mathbf{C}^{bk}(\mathbf{b})\right)\cdot\nabla\mathbf{B}$$
$$\frac{D}{Dt}\left\{\upsilon(\mathbf{Z_a}-\nabla\beta\times\nabla s)\cdot\nabla\mathbf{B}\right\}-\upsilon(\mathbf{Z_a}-\nabla\beta\times\nabla s)\cdot\nabla\left(\frac{D\mathbf{B}}{Dt}\right)=\upsilon\left(\nabla\times\mathbf{C}^{bk}(\mathbf{b})\right)\cdot\nabla\mathbf{B} . \tag{4.10}$$

The first equation in (4.9) can be rewritten if we observe that $\nabla\cdot\mathbf{Z_a}=0,\quad \nabla\cdot\left\{\nabla\times\mathbf{C}^{bt}(\mathbf{b})\right\}=0$. Then:

$$\frac{D_3(\mathbf{Z_a}\cdot\nabla\mathbf{B})}{Dt}=\nabla\cdot\left\{\mathbf{Z_a}\frac{D\mathbf{B}}{Dt}+\left(\nabla\times\mathbf{C}^{bt}(\mathbf{b})\right)\mathbf{B}\right\}. \tag{4.11}$$

Integrating, and applying GAUSS´s theorem, the integral version is:
m

$$\frac{D}{Dt}\left\{\iiint_{V(t)}d\tau\,(\mathbf{Z_a}\cdot\nabla\mathbf{B})\right\}=\oiint do\cdot\left\{\mathbf{Z_a}\frac{D\mathbf{B}}{Dt}+\left(\nabla\times\mathbf{C}^{bt}(\mathbf{b})\right)\mathbf{B}\right\}$$

Ertel´s best-known vorticity theorems are members of the previous equations. ERTEL always considers an ideal fluid, so the right-hand side of the second member of (4.5) as well as the one of (4.9) is $\mathbf{N}(p,\upsilon)$. Then, the theorem from the year 1955 (ERTEL, **132**, **133**) can be written as:

$$\frac{D(\upsilon \mathbf{Z_a} \cdot \mathbf{A})}{Dt} - \upsilon \mathbf{Z_a} \cdot \frac{D_1 \mathbf{A}}{Dt} = \upsilon \mathbf{N}(p,\upsilon) \cdot \mathbf{A}. \tag{4.12}$$

The famous theorem from the year 1942 (Ertel **92**, **93**, **95**, **96**) is

$$\frac{D(\upsilon \mathbf{Z_a} \cdot \nabla \mathbf{B})}{Dt} - \upsilon \mathbf{Z_a} \cdot \nabla\left(\frac{D\mathbf{B}}{Dt}\right) = \upsilon \mathbf{N}(p,\upsilon) \cdot \nabla \mathbf{B}. \tag{4.13}$$

If $\mathbf{B} = s$, and if the motion is adiabatic, then ERTEL´s potential vorticity $\Pi_\mathbf{a} = \upsilon \mathbf{Z_a} \cdot \nabla s$ is a material invariant: $D\Pi_\mathbf{a}/Dt = 0$. From (4.9) the same result is obtained. Except for the Eulerian equations of motion, this is the only result from geophysical fluid dynamics that is widely used in meteorology. The "PV-era" in meteorology started many years after ERTEL´s death. Finally, we point out that the second member of equations (4.7) is the ERTEL-ROSSBY theorem (ERTEL et al. **120**, **121**).

Equations (4.2), (4.5)/(4.6), and (4.9)/(4.10), represent a group of general vorticity theorems; they have their origins in very simple identities in which the physical input came from the Eulerian equations of motion. These entered in a variety of versions depending on the circumstances chosen. All of these can now be written in terms of our "macroscopic" material derivatives. The first members of equations (4.5), (4.6), (4.7), (4.9), and (4.10) are appropriate for integration over material volumes, leading to "macroscopic" conservation principles, the second ones are for exploring more material invariants of the $D/Dt = 0$ type.

With respect to a further important paper of ERTEL (ERTEL, **152**, **153**), an identity connected with *generalized helicity* is obtained from (4.3)

$$\frac{D_3(\mathbf{a} \cdot \nabla \times \mathbf{a})}{Dt} = (\nabla \times \mathbf{a}) \cdot \frac{D_1 \mathbf{a}}{Dt} + \frac{D_2(\nabla \times \mathbf{a})}{Dt} \cdot \mathbf{a} = (\nabla \times \mathbf{a}) \cdot \frac{D_1 \mathbf{a}}{Dt} + \left(\nabla \times \frac{D_1 \mathbf{a}}{Dt}\right) \cdot \mathbf{a}$$

$$\frac{D_3(\mathbf{a} \cdot \nabla \times \mathbf{a})}{Dt} = (\nabla \times \mathbf{a}) \cdot \frac{D_1 \mathbf{a}}{Dt} + \left(\nabla \times \frac{D_1 \mathbf{a}}{Dt}\right) \cdot \mathbf{a}. \tag{4.14}$$

In order to relate the work of ERTEL to the result (4.14), we introduce curvilinear and time dependent *MONGE potentials* $\theta^1, \theta^2, \theta^3$. Then vector $\mathbf{a}$ can be represented as:

$$\mathbf{a} = \nabla\theta^1 + \theta^2 \nabla\theta^3. \tag{4.15}$$

Now, $\nabla \times \mathbf{a} = \nabla\theta^2 \times \nabla\theta^3$, and $\mathbf{a} \cdot \nabla \times \mathbf{a} = \nabla\theta^1 \cdot \nabla\theta^2 \times \nabla\theta^3 = J$. The Jacobian *J* is a generalized helicity. If we introduce this into (4.14), we need:

$$\frac{D_1 \mathbf{a}}{Dt} = \nabla\left(\frac{D\theta^1}{Dt}\right) + \theta^2 \nabla\left(\frac{D\theta^3}{Dt}\right) + \frac{D\theta^2}{Dt} \nabla\theta^3.$$

ERTEL´s result (ERTEL, **152**, **153**) can now be formulated as:

$$\begin{aligned}\frac{D_3 J}{Dt} &= \rho \frac{D\upsilon J}{Dt} = \frac{\partial J}{\partial t} + \nabla \cdot (J\mathbf{V_a}) = (\nabla \times \mathbf{a}) \cdot \frac{D_1 \mathbf{a}}{Dt} + \left(\nabla \times \frac{D_1 \mathbf{a}}{Dt}\right) \cdot \mathbf{a} = \\ &= \left(\nabla\theta^2 \times \nabla\theta^3\right) \cdot \nabla\left(\frac{D\theta^1}{Dt}\right) + \left(\nabla\theta^3 \times \nabla\theta^1\right) \cdot \nabla\left(\frac{D\theta^2}{Dt}\right) + \left(\nabla\theta^1 \times \nabla\theta^2\right) \cdot \nabla\left(\frac{D\theta^3}{Dt}\right) = \nabla \cdot \left\{\mathbf{\Psi} \cdot \frac{D\boldsymbol{\theta}}{Dt}\right\}.\end{aligned} \tag{4.16}$$

In (4.16) the adjoint $\mathbf{\Psi}$ of $\mathbf{\Phi} = \nabla\boldsymbol{\theta}$ was introduced (BRAND, 1962):

$$\mathbf{\Psi} = \left(\nabla\theta^2 \times \nabla\theta^3\right)\mathbf{i} + \left(\nabla\theta^3 \times \nabla\theta^1\right)\mathbf{j} + \left(\nabla\theta^1 \times \nabla\theta^2\right)\mathbf{k}, \quad \text{with} \quad \nabla \cdot \mathbf{\Psi} = 0.$$

Integrating (4.16) over a material volume and applying GAUSS´s theorem we obtain the interesting result

$$\frac{D}{Dt}\left\{\iiint_{V(t)} d\tau\, J\right\} = \oiint_{O(t)} d\mathbf{o}\cdot\left\{\Psi\cdot\frac{D\theta}{Dt}\right\}.$$

If $\mathbf{a}$ is identically with the absolute velocity vector $\mathbf{V_a}$, then $J = \mathbf{V_a}\cdot\mathbf{Z_a} = h_a$ is *absolute helicity* (a scalar) in fluid dynamics. Therefore, equation (4.16) is the helicity equation that depends on the motion of the coordinates.

If we know (zero-dimensional) material invariants $\theta^1, \theta^2, \theta^3$, a number of interesting conclusions can be drawn. First, we have:

$$\frac{D_3 J}{Dt} = \frac{DJ}{Dt} + J\nabla\cdot\mathbf{V_a} = -J^2\left\{\frac{D}{Dt}\left(\frac{1}{J}\right) - \left(\frac{1}{J}\right)\nabla\cdot\mathbf{V_a}\right\} = 0.$$

So, we find the divergence of the velocity vector as

$$\nabla\cdot\mathbf{V_a} = \nabla\cdot\mathbf{V} = \frac{1}{(1/J)}\frac{D(1/J)}{Dt}. \tag{4.17}$$

Furthermore, we obtain a number of conservation principles from $D_3 J/Dt = 0,\quad D(\upsilon J)/Dt = 0$. Ertel (**197**) has shown that the maximum number of independent invariants in fluid dynamics, (i.e., those for which $D/Dt = 0$), is equal to the number of dimensions in the reference space. The Lagrangian coordinates a^1, a^2, a^3 are such invariants: $D\mathbf{a}/Dt = 0$. Let be $J = J(a^1, a^2, a^3)$, and using the equation of continuity in Lagrangian coordinates, $\upsilon J(a^1, a^2, a^3) = \upsilon|_{t_0}$, then

$$\upsilon\frac{D_3 J}{Dt} = \frac{D}{Dt}\left\{\upsilon J(a^1, a^2, a^3)\right\} = \frac{D\upsilon|_{t_0}}{Dt} = 0.$$

We see that $\upsilon|_{t_0}$ is a second invariant. For adiabatic motion, entropy *s* is an invariant ($Ds/Dt = 0$). When the conditions for which Ertel´s potential vorticity are fulfilled, PV=Π_a is another invariant ($D\Pi_a/Dt = 0$). Employing the three collected invariants $\upsilon|_{t_0}$, s, and Π_a, $\upsilon J\left(\upsilon|_{t_0}, s, \Pi_a\right) = \upsilon\nabla\upsilon|_{t_0}\cdot\nabla s\times\nabla\Pi_a$ is a combined additional invariant. In (4.7) we got at least two more invariants. For ideal fluids and barotropic flows, we had the *ERTEL- ROSSBY invariant* $\upsilon\mathbf{Z_a}\cdot\left(\mathbf{V_a} - \nabla W_a^{bk}\right)$, and for the same frictionless fluid and adiabatic flow, the invariant is $\upsilon\left(\mathbf{Z_a} - \nabla\beta\times\nabla s\right)\cdot\left(\mathbf{V_a} - \beta\nabla s - \nabla W_a^{bk}\right)$. We would be able to continue the search for more invariants of that kind.

The listed identities, together with the integrated versions, are the basis for a very large number of vorticity and circulation theorems. This is true not only in fluid dynamics, but in other branches of theoretical physics as well, when motions of continua are involved. Physics enter in these identities if we know for a vector $\mathbf{a}$ its physical equation $D_2\mathbf{a}/Dt = \mathbf{C}$, and if additionally we choose an arbitrary function (scalar, vector, tensor) for which physical equations like $D\mathbf{B}/Dt = \mathbf{F}_0$ or $D_1\mathbf{B}/Dt = \mathbf{F}_1$ are known. Comfortable conditions are those in which $\mathbf{F}_0$ or $\mathbf{F}_1 = 0$. The resulting equations constitute generalized physical principles. In meteorology, such new principles were formulated first by ERTEL (ERTEL, **92**, **93**, **95**, **96**, **120**, **121**, **132**, **133**).

5. Material "conservation" of the circulation of eddy velocities in atmospheric turbulence

In one of his early papers (ERTEL, **13**), ERTEL showed that under certain conditions the circulation of eddy velocities around a closed material contour could be an invariant material property. In his paper, he assumed *incompressibility, steady state conditions of the mean fields, and a non-rotating coordinate system*. We will generalize his result, applying the mathematical techniques developed above.

The material rate of change with time of the circulation of eddy velocities $\mathbf{V}''$ is represented by

$$\frac{D}{Dt}\left[\oint_{C(t)} d\mathbf{s}\cdot\mathbf{V}''\right].$$

In accordance with REYNOLDS´s techniques, the velocity is split up as $\mathbf{V_a}=\hat{\mathbf{V}}_\mathbf{a}+\mathbf{V}''$. The hat operator denotes density weighted REYNOLDS´s averaging (including the appropriate rules).

The equation of motion in momentum version is given by

$$\rho\frac{D_a\mathbf{V}_a}{\mathrm{Dt}}=-\rho\nabla\phi^{(A)}+\nabla\cdot\mathbf{F}_{tot},\quad \text{with}\quad \mathbf{F}_{tot}=-p\mathbf{I}+\mathbf{F}. \tag{5.1}$$

Density weighted averaging yields:

$$\overline{\rho}\frac{\hat{D}_a\hat{\mathbf{V}}_a}{\mathrm{Dt}}=-\overline{\rho}\nabla\phi^{(A)}+\nabla\cdot\left(\overline{\mathbf{F}}_{tot}+\overline{\mathbf{R}}\right),\quad \text{with}\quad \overline{\mathbf{R}}\text{: Reynolds' stress tensor}. \tag{5.2}$$

Correspondingly,

$$\frac{D_{1,a}\mathbf{V}_a}{\mathrm{Dt}}=\frac{D_1\mathbf{V}_a}{\mathrm{Dt}}=\nabla\left(\frac{\mathbf{V}_a^2}{2}-\phi^{(A)}\right)+\upsilon\nabla\cdot\mathbf{F}_{tot}, \tag{5.1a}$$

$$\frac{\hat{D}_{1,a}\hat{\mathbf{V}}_a}{\mathrm{Dt}}=\frac{\hat{D}_1\hat{\mathbf{V}}_a}{\mathrm{Dt}}=\nabla\left(\frac{\hat{\mathbf{V}}_a^2}{2}-\phi^{(A)}\right)+\overline{\upsilon}\nabla\cdot\left(\overline{\mathbf{F}}_{tot}+\overline{\mathbf{R}}\right). \tag{5.2a}$$

Here $D/Dt=\partial/\partial t+\mathbf{V}\cdot\nabla$ and $\hat{D}/Dt=\partial/\partial t+\hat{\mathbf{V}}\cdot\nabla$.

Applying the curl operation on (5.1a) and (5.2a) as before, we get the vorticity equations for both, the non-averaged and the averaged equations of motion:

$$\frac{D_2\mathbf{Z}_a}{\mathrm{Dt}}=\nabla\times\left(\upsilon\nabla\cdot\mathbf{F}_{tot}\right) \tag{5.3}$$

$$\frac{\hat{D}_2\hat{\mathbf{Z}}_a}{\mathrm{Dt}}=\nabla\times\left[\overline{\upsilon}\nabla\cdot\left(\overline{\mathbf{F}}_{tot}+\overline{\mathbf{R}}\right)\right]. \tag{5.4}$$

Integrating the vorticity equations (5.3) and (5.4) over a material surface area of the fluid, and applying STOKES´s theorem, with (1.2) we get:

$$\frac{D}{Dt}\left\{\oint_{C(t)} d\mathbf{s}\cdot\mathbf{V}_a\right\}=\frac{D}{Dt}\left\{\oint_{C(t)} d\mathbf{s}\cdot\hat{\mathbf{V}}_a\right\}+\frac{D}{Dt}\left\{\oint_{C(t)} d\mathbf{s}\cdot\mathbf{V}''\right\}=\oint_{C(t)} d\mathbf{s}\cdot\left(\upsilon\nabla\cdot\mathbf{F}_{tot}\right), \tag{5.5}$$

$$\frac{\hat{D}}{Dt}\left\{\oint_{C(t)} d\mathbf{s}\cdot\hat{\mathbf{V}}_a\right\}=\oint_{C(t)} d\mathbf{s}\cdot\left[\overline{\upsilon}\nabla\cdot\left(\overline{\mathbf{F}}_{tot}+\overline{\mathbf{R}}\right)\right]. \tag{5.6}$$

We now write (5.5) as

$$\frac{D}{Dt}\left\{\oint_{C(t)} d\mathbf{s}\cdot\mathbf{V}''\right\} = -\frac{D}{Dt}\left\{\oint_{C(t)} d\mathbf{s}\cdot\hat{\mathbf{V}}_a\right\} + \oint_{C(t)} d\mathbf{s}\cdot\left(\upsilon\nabla\cdot\mathbf{F}_{tot}\right) = -\frac{D}{Dt}\left\{\iint_{F(t)} d\mathbf{f}\cdot\hat{\mathbf{Z}}_a\right\} + \oint_{C(t)} d\mathbf{s}\cdot\left(\upsilon\nabla\cdot\mathbf{F}_{tot}\right). \quad (5.7)$$

Eq. (1.2) yields for the first term on the right:

$$\frac{D}{Dt}\left\{\iint_{F(t)} d\mathbf{f}\cdot\hat{\mathbf{Z}}_a\right\} = \iint_{F(t)} d\mathbf{f}\cdot\frac{D_2\hat{\mathbf{Z}}_a}{Dt}. \quad (5.8)$$

The following relationship is valid for any vector or tensor $\mathbf{A}$, thus for $\hat{\mathbf{Z}}_a$ with $\nabla\cdot\hat{\mathbf{Z}}_a = 0$:

$$\frac{D_2\mathbf{A}}{Dt} = \frac{\hat{D}_2\mathbf{A}}{Dt} - \nabla\times\left(\mathbf{V}''\times\mathbf{A}\right) + \mathbf{V}''\left(\nabla\cdot\mathbf{A}\right),$$

and consequently:

$$\frac{D_2\hat{\mathbf{Z}}_a}{Dt} = \frac{\hat{D}_2\hat{\mathbf{Z}}_a}{Dt} - \nabla\times\left(\mathbf{V}''\times\hat{\mathbf{Z}}_a\right). \quad (5.9)$$

The final result is obtained if (5.4), (5.6), (5.8), and (5.9) are introduced in (5.7):

$$\frac{D}{Dt}\left\{\oint_{C(t)} d\mathbf{s}\cdot\mathbf{V}''\right\} = \oint_{C(t)} d\mathbf{s}\cdot\left(\mathbf{V}''\times\hat{\mathbf{Z}}_a\right) + \oint_{C(t)} d\mathbf{s}\cdot\left\{\upsilon\nabla\cdot\mathbf{F}_{tot} - \bar{\upsilon}\nabla\cdot\left(\overline{\mathbf{F}}_{tot} + \overline{\mathbf{R}}\right)\right\}. \quad (5.10)$$

This equation (5.10) has an important implication. If the material line of integration $d\mathbf{s}$ is chosen such that it coincides with a closed material line of the mean absolute vorticity vector $\hat{\mathbf{Z}}_a$, then the first term on the right-hand side vanishes. The question arises whether closed material lines of the mean absolute vorticity vector $\hat{\mathbf{Z}}_a$ do exist within the atmosphere (boundary layer theory, jet stream?).

If the fluid is frictionless (= ideal) and barotropic, the last term on the right-hand side is written as:

$$\upsilon\nabla\cdot\mathbf{F}_{tot} - \bar{\upsilon}\nabla\cdot\left(\overline{\mathbf{F}}_{tot} + \overline{\mathbf{R}}\right) = -\bar{\upsilon}\nabla\cdot\overline{\mathbf{R}}.$$

The result reads as:

$$\frac{D}{Dt}\left\{\oint_{C(t)} d\mathbf{s}\cdot\mathbf{V}''\right\} = \oint_{C(t)} d\mathbf{s}\cdot\left(\mathbf{V}''\times\hat{\mathbf{Z}}_a\right) - \oint_{C(t)} d\mathbf{s}\cdot\left\{\bar{\upsilon}\nabla\cdot\overline{\mathbf{R}}\right\}. \quad (5.11)$$

If the material line of integration $d\mathbf{s}$ is chosen as was discussed in connection with (5.10), then we get the result that the left-hand side depends only on the slowly varying mean field of REYNOLDS´ stress. This result here may be interpreted as almost material conservation of the circulation of eddy velocities in atmospheric turbulence under such conditions. In ERTEL´s paper, the mean field is a steady-state one. So, he got the result that the circulation in (5.11) is a material invariant with time. This certainly was a remarkable result in theory of turbulence obtained by the student Hans ERTEL.

Further applications of the formalism developed here are possible in the fields of electrodynamics of fluids and in radiation hydrodynamics; they will be considered in a later study.

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A Appendix 1: Lagrangian fluid dynamics with application in parts 1., and 3.

In Lagrangian fluid dynamics, the initial position vector is $\mathbf{a} = \mathbf{i}_n a^n$. Here, the initial coordinates a^1, a^2, a^3 are material invariants, i. e. $D\mathbf{a}/Dt = 0$. The coordinates x^1, x^2, x^3 of particles at later times define the position vector **R**. The trajectory of a particle with initial position $\mathbf{a}$ is given by $\mathbf{R} = \mathbf{R}(\mathbf{a}, t)$.

We define a covariant vector basis $\mathbf{g}_k$ and a contravariant vector basis $\mathbf{g}^k$ by

$$
\begin{aligned}
&\mathbf{g}_k = \frac{\partial \mathbf{R}}{\partial a^k}, \quad \mathbf{\Theta} = \mathbf{i}^m \mathbf{g}_m = \frac{\partial \mathbf{R}}{\partial \mathbf{a}}, \quad \sqrt{g} = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3, \\
&\mathbf{g}^1 = \frac{\mathbf{g}_2 \times \mathbf{g}_3}{\sqrt{g}}, \quad \mathbf{g}^2 = \frac{\mathbf{g}_3 \times \mathbf{g}_1}{\sqrt{g}}, \quad \mathbf{g}^3 = \frac{\mathbf{g}_1 \times \mathbf{g}_2}{\sqrt{g}}, \quad \mathbf{\Theta}^{-1} = \mathbf{g}^m \mathbf{i}_m, \quad \frac{1}{\sqrt{g}} = \mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3, \\
&\mathbf{g}_k \cdot \mathbf{g}^l = \delta_k^l, \quad , \quad \mathbf{I} = \mathbf{i}^m \mathbf{i}_m = \mathbf{g}^m \mathbf{g}_m = \mathbf{\Theta}^{-1} \cdot \mathbf{\Theta}.
\end{aligned}
\tag{A.1}
$$

The unit dyadic is **I**, KRONECKER´s symbol is δ_k^l. $\mathbf{\Theta}$ is the deformation tensor and $\mathbf{\Theta}^{-1}$ is the reciprocal. Additionally, the summation convention is applied: If in a product two indices are equal, it is understood that the sum is taken from 1 to 3.

We transform the ∇ - operator from the Eulerian frame to the Lagrangian one and vice versa by

$$
\nabla = \frac{\partial}{\partial \mathbf{R}} = \mathbf{g}^m \frac{\partial}{\partial a^m} = \mathbf{\Theta}^{-1} \cdot \frac{\partial}{\partial \mathbf{a}}, \qquad \frac{\partial}{\partial \mathbf{a}} = \mathbf{i}^m \frac{\partial}{\partial a^m} = \mathbf{\Theta} \cdot \nabla. \tag{A.2}
$$

In Lagrangian fluid dynamics, the material derivative D/Dt reads $D/Dt = \partial/\partial t|_a$. Then, $\partial/\partial a^k \left(\partial/\partial t|_a\right) = \partial/\partial t|_a \left(\partial/\partial a^k\right)$. Instead, we write $\partial/\partial a^k (D/Dt) = D/Dt\left(\partial/\partial a^k\right)$.

First, we differentiate the unit dyadic

$$
\frac{D\mathbf{I}}{Dt} = \frac{D\mathbf{g}^m}{Dt}\mathbf{g}_m + \mathbf{g}^m \frac{D\mathbf{g}_m}{Dt} = \frac{D\mathbf{\Theta}^{-1}}{Dt} \cdot \mathbf{\Theta} + \mathbf{\Theta}^{-1} \cdot \frac{D\mathbf{\Theta}}{Dt} = 0. \tag{A.3}
$$

Then with $\mathbf{V} = D\mathbf{R}/Dt$, and $\nabla \mathbf{V} = \dfrac{\partial \mathbf{V}}{\partial \mathbf{R}} = \mathbf{g}^m \dfrac{\partial \mathbf{V}}{\partial a^m} = \mathbf{g}^m \dfrac{D\mathbf{g}_m}{Dt} = -\dfrac{D\mathbf{g}^m}{Dt}\mathbf{g}_m$,

$$
\mathbf{g}_k \cdot \nabla \mathbf{V} = \frac{D\mathbf{g}_k}{Dt}, \quad \mathbf{\Theta} \cdot \nabla \mathbf{V} = \frac{D\mathbf{\Theta}}{Dt}, \qquad (\nabla \mathbf{V}) \cdot \mathbf{g}^k = -\frac{D\mathbf{g}^k}{Dt}, \quad (\nabla \mathbf{V}) \cdot \mathbf{\Theta}^{-1} = -\frac{D\mathbf{\Theta}^{-1}}{Dt}. \tag{A.4}
$$

In Lagrangian version, a material line element, an element of a surface area, and an element of a volume are:

$$
d\mathbf{s} = da^m \mathbf{g}_m, \qquad d\mathbf{f} = da^1 da^2 \mathbf{g}_1 \times \mathbf{g}_2 = da^1 da^2 \sqrt{g}\, \mathbf{g}^3, \qquad d\tau(R) = d\tau(a)\sqrt{g}. \tag{A.5}
$$

Here, we avoided the use of the permutation symbol ε^{klm} in $d\mathbf{f}$ by selecting a special surface area element on $a^3 = \text{const}$.

We now use the definitions of the operators D_1/Dt, D_2/Dt, D_3/Dt and transform them from "Euler to Lagrange":

$$
\mathbf{g}_k \cdot \frac{D_1 \mathbf{A}}{Dt} = \frac{D}{Dt}(\mathbf{g}_k \cdot \mathbf{A}) - \left\{\frac{D\mathbf{g}_k}{Dt} - \mathbf{g}_k \cdot \nabla \mathbf{V}\right\} \cdot \mathbf{A} = \frac{D}{Dt}(\mathbf{g}_k \cdot \mathbf{A}), \tag{A.6}
$$

$$
\begin{aligned}
&\sqrt{g}\, \mathbf{g}^k \cdot \frac{D_2 \mathbf{A}}{Dt} = \frac{D}{Dt}\left(\sqrt{g}\, \mathbf{g}^k \cdot \mathbf{A}\right) - \left\{\frac{D\sqrt{g}}{Dt}\mathbf{g}^k + \sqrt{g}\frac{D\mathbf{g}^k}{Dt} - \sqrt{g}(\nabla \cdot \mathbf{V})\mathbf{g}^k + \sqrt{g}\mathbf{g}^k \cdot (\mathbf{V}\nabla)\right\} \cdot \mathbf{A} = \\
&= \frac{D}{Dt}\left(\sqrt{g}\, \mathbf{g}^k \cdot \mathbf{A}\right) - \left\{\frac{D\sqrt{g}}{Dt}\mathbf{g}^k + \sqrt{g}\frac{D\mathbf{g}^k}{Dt} - \sqrt{g}\frac{1}{\sqrt{g}}\frac{D\sqrt{g}}{Dt}\mathbf{g}^k - \sqrt{g}\frac{D\mathbf{g}^k}{Dt}\right\} \cdot \mathbf{A} = \frac{D}{Dt}\left(\sqrt{g}\, \mathbf{g}^k \cdot \mathbf{A}\right)
\end{aligned}
, \tag{A.7}
$$

$$\sqrt{g}\,\frac{D_3\mathbf{A}}{Dt} = \frac{D}{Dt}\left(\sqrt{g}\ \mathbf{A}\right) - \left\{\frac{D\sqrt{g}}{Dt} - \sqrt{g}(\nabla\cdot\mathbf{V})\right\}\mathbf{A} = \frac{D}{Dt}\left(\sqrt{g}\ \mathbf{A}\right). \qquad \text{(A.8)}$$

These formulae are valid with regard to both inertial and relative frames, and they provide for a connection to ERTEL´s work in **200**, **209**, **260**, **124**. In inertial frames, we multiply with the *da´*s according to (A.5) and integrate over the corresponding material structures. Observing that now $\int D/Dt = D/Dt\int$, with (A.5) we get the result that the equations (1.1) to (1.3) are valid. In addition, with all foregoing results, we find directly

$$\frac{Dd\mathbf{s}}{Dt} = d\mathbf{s}\cdot\nabla\mathbf{V}, \quad \frac{Dd\mathbf{f}}{Dt} = d\mathbf{f}\cdot\left\{(\nabla\cdot\mathbf{V})\mathbf{I} - (\mathbf{V}\nabla)\right\}, \quad \frac{Dd\tau}{Dt} = d\tau(\nabla\cdot\mathbf{V}). \qquad \text{(A.9)}$$

We summarize: Transforming the operators D_1/Dt, $D_2/Dt, D_3/Dt$ into Lagrangian coordinates, we have

$$\mathbf{g}_k\cdot\frac{D_1\mathbf{A}}{Dt} = \frac{D}{Dt}(\mathbf{g}_k\cdot\mathbf{A}), \qquad \mathbf{\Theta}\cdot\frac{D_1\mathbf{A}}{Dt} = \frac{D}{Dt}(\mathbf{\Theta}\cdot\mathbf{A}), \qquad \text{(A.6a)}$$

$$\sqrt{g}\ \mathbf{g}^k\cdot\frac{D_2\mathbf{A}}{Dt} = \frac{D}{Dt}\left(\sqrt{g}\ \mathbf{g}^k\cdot\mathbf{A}\right), \qquad \sqrt{g}\left(\mathbf{\Theta}^{-1}\right)^T\cdot\frac{D_2\mathbf{A}}{Dt} = \frac{D}{Dt}\left(\sqrt{g}\left(\mathbf{\Theta}^{-1}\right)^T\cdot\mathbf{A}\right), \qquad \text{(A.7a)}$$

$$\sqrt{g}\,\frac{D_3\mathbf{A}}{Dt} = \frac{D}{Dt}\left(\sqrt{g}\ \mathbf{A}\right). \qquad \text{(A.8)}$$

It was mentioned that these formulae are valid with regard to both inertial and relative frames. In inertial frames, we attach the index "a" to all symbols where this is appropriate. For example, it is $\mathbf{\Theta}_{\mathbf{a}} = \partial\mathbf{R}_{\mathbf{a}}/\partial\mathbf{a}$, and equation (A.6a) read as $\mathbf{\Theta}_{\mathbf{a}}\cdot D_{1\mathbf{a}}\mathbf{A}/Dt = D_{\mathbf{a}}/Dt\left(\mathbf{\Theta}_{\mathbf{a}}\cdot\mathbf{A}\right)$. Due to the invariance of the operator D_1/Dt with respect to solid rotations, multiplication of equations (3.4) and (3.9) by $\mathbf{\Theta}_{\mathbf{a}}$ leads to the Lagrangian version of these equations (see also ERTEL, **200**, **209**, **260**)

$$\frac{D_{\mathbf{a}}}{Dt}\left(\mathbf{\Theta}_{\mathbf{a}}\cdot\mathbf{V}_{\mathbf{a}} - \frac{\partial W_{\mathbf{a}}^{bt}}{\partial\mathbf{a}}\right) = -\upsilon\frac{\partial p}{\partial\mathbf{a}} + \frac{\partial P}{\partial\mathbf{a}} + \upsilon\frac{\partial}{\partial\mathbf{a}}\cdot\mathbf{F}, \qquad \text{(A.10)}$$

$$\frac{D_{\mathbf{a}}}{Dt}\left(\mathbf{\Theta}_{\mathbf{a}}\cdot\mathbf{V}_{\mathbf{a}} - \beta\frac{\partial s}{\partial\mathbf{a}} - \frac{\partial W_{\mathbf{a}}^{bk}}{\partial\mathbf{a}}\right) = -\beta\frac{\partial}{\partial\mathbf{a}}\left(\frac{Ds}{Dt}\right) + \upsilon\frac{\partial}{\partial\mathbf{a}}\cdot\mathbf{F}. \qquad \text{(A.11)}$$

These equations are integrated along a single trajectory of a particle that had started at time $t_0(\mathbf{a})$ in position $\mathbf{R}_0 = \mathbf{R}(\mathbf{a}, t_0(\mathbf{a}))$ and that reached position $\mathbf{R} = \mathbf{R}(\mathbf{a}, t)$ in space at time *t*. The running time then is $t - t_0(\mathbf{a})$. First, we consider (A.10) only and integrate

$$\mathbf{\Theta}_{\mathbf{a}}\cdot\mathbf{V}_{\mathbf{a}} - \frac{\partial W_{\mathbf{a}}^{bt}}{\partial\mathbf{a}} - \left(\mathbf{\Theta}_{\mathbf{a}}\cdot\mathbf{V}_{\mathbf{a}} - \frac{\partial W_{\mathbf{a}}^{bt}}{\partial\mathbf{a}}\right)\Bigg|_{t_0} = \int_{t_0(\mathbf{a})}^{t} dt\left\{-\upsilon\frac{\partial p}{\partial\mathbf{a}} + \frac{\partial P}{\partial\mathbf{a}} + \upsilon\frac{\partial}{\partial\mathbf{a}}\cdot\mathbf{F}\right\}.$$

ERTEL, in his paper (ERTEL, 1952 (**124**)), brilliantly derived an equivalent expression for the following relationship (originally for barotropic flow, but actually valid in general)

$$\left(\mathbf{\Theta}_{\mathbf{a}}\cdot\mathbf{V}_{\mathbf{a}} - \frac{\partial W_{\mathbf{a}}^{bt}}{\partial\mathbf{a}}\right)\Bigg|_{t_0} = -H_{\mathbf{a}}^{bt}\Big|_{t_0}\frac{\partial t_0(\mathbf{a})}{\partial\mathbf{a}} = H_{\mathbf{a}}^{bt}\Big|_{t_0}\frac{\partial(t - t_0(\mathbf{a}))}{\partial\mathbf{a}}. \qquad \text{(A.12)}$$

Here $H_{\mathbf{a}}^{bt} = \mathbf{V}_{\mathbf{a}}^2/2 + \phi^{(A)} + P)$ is total energy per unit mass. Now WEBER´s transformation, in Lagrangian form is given by:

$$\mathbf{\Theta_a} \cdot \mathbf{V_a} - \frac{\partial W_\mathbf{a}^{bt}}{\partial \mathbf{a}} - H_\mathbf{a}^{bt}\Big|_{t_0} \frac{\partial(t - t_0(\mathbf{a}))}{\partial \mathbf{a}} = \int_{t_0(\mathbf{a})}^{t} dt \left\{ -\upsilon \frac{\partial p}{\partial \mathbf{a}} + \frac{\partial P}{\partial \mathbf{a}} + \upsilon \frac{\partial}{\partial \mathbf{a}} \cdot \mathbf{F} \right\}. \qquad \text{(A.13)}$$

We return to the Eulerian version of (A.13) by multiplying this equation with $\mathbf{\Theta_a^{-1}}$. Then, we get the equivalent of ClLEBSCH´s transformation as

$$\mathbf{V_a} = \nabla W_\mathbf{a}^{bt} + H_\mathbf{a}^{bt}\Big|_{t_0} \nabla(t - t_0(\mathbf{a})) + \mathbf{\Theta_a^{-1}} \cdot \int_{t_0(\mathbf{a})}^{t} dt\, \mathbf{\Theta_a} \cdot \left\{ -\upsilon \nabla p + \nabla P + \upsilon \nabla \cdot \mathbf{F} \right\}. \qquad \text{(A.14)}$$

Ertel considered steady-state barotropic flow of ideal fluids. In that steady-state case, total energy is a material invariant, $H_\mathbf{a}^{bt}\Big|_{t_0} = H_\mathbf{a}^{bt}$. His theorem in (**124**) then reads:

$$\mathbf{V_a} = \nabla W_\mathbf{a}^{bt} + H_\mathbf{a}^{bt} \nabla(t - t_0(\mathbf{a})). \qquad \text{(A.15)}$$

The proof of uniqueness of equation (A.15) was given by FORTAK (FORTAK, 1954).

The case that is applicable to adiabatic motion with $\beta(t_0) = 0$ is given by

$$\mathbf{V_a} = \nabla W_\mathbf{a}^{bk} + \beta \nabla s + H_\mathbf{a}^{bk}\Big|_{t_0} \nabla(t - t_0(\mathbf{a})) + \mathbf{\Theta_a^{-1}} \cdot \int_{t_0(\mathbf{\alpha})}^{t} dt\, \mathbf{\Theta_a} \cdot \left\{ -\beta \nabla \frac{Ds}{Dt} + \upsilon \nabla \cdot \mathbf{F} \right\}. \qquad \text{(A.16)}$$

Here total energy is $H_\mathbf{a}^{bk} = \mathbf{V_a^2}/2 + \phi^{(A)} + h$.

For steady-state adiabatic flows of ideal fluids, we have

$$\mathbf{V_a} = \nabla W_\mathbf{a}^{bk} + \beta \nabla s + H_\mathbf{a}^{bk} \nabla(t - t_0(\mathbf{a})). \qquad \text{(A.17)}$$

B Appendix 2: Proof of equations (1.4)

Equations (1.4) can be proved directly by vector calculations. We prefer the following method where equations (1.1 to 1.3) are used together with the integral theorems of GAUSS and STOKES. Integrating the members of (1.4) with respect to a material line, an area, and a volume, we have the equations to be proved as

$$\int_{S_1}^{S_2} d\mathbf{s} \cdot \nabla\left(\frac{D\mathbf{A}}{Dt}\right) = \int_{S_1}^{S_2} d\mathbf{s} \cdot \frac{D_1}{Dt}(\nabla\mathbf{A}), \quad \iint_{F(t)} d\mathbf{f} \cdot \nabla \times \left(\frac{D_1\mathbf{A}}{Dt}\right) = \iint_{F(t)} d\mathbf{f} \cdot \frac{D_2}{Dt}(\nabla \times \mathbf{A}),$$

$$\iiint_{V(t)} d\tau\, \nabla \cdot \left(\frac{D_2\mathbf{A}}{Dt}\right) = \iiint_{V(t)} d\tau\, \frac{D_3}{Dt}(\nabla \cdot \mathbf{A}).$$

Now we proceed as follows:

$$\int_{S_1}^{S_2} d\mathbf{s} \cdot \nabla\left(\frac{D\mathbf{A}}{Dt}\right) = \int_{S_1}^{S_2} d\left(\frac{D\mathbf{A}}{Dt}\right) \qquad \int_{S_1}^{S_2} d\mathbf{s} \cdot \frac{D_1}{Dt}(\nabla\mathbf{A}) = \frac{D}{Dt}\int_{S_1}^{S_2} d\mathbf{s} \cdot (\nabla\mathbf{A}) = \frac{D}{Dt}\int_{S_1}^{S_2} d\mathbf{A} = \int_{S_1}^{S_2} d\left(\frac{D\mathbf{A}}{Dt}\right)$$

$$\iint_{F(t)} d\mathbf{f} \cdot \nabla \times \left(\frac{D_1\mathbf{A}}{Dt}\right) = \oint_{C(F)} d\mathbf{s} \cdot \left(\frac{D_1\mathbf{A}}{Dt}\right) = \frac{D}{Dt} \oint_{C(F)} d\mathbf{s} \cdot \mathbf{A} =$$

$$= \iint_{F(t)} d\mathbf{f} \cdot \frac{D_2}{Dt}(\nabla \times \mathbf{A}) = \frac{D}{Dt}\iint_{F(t)} d\mathbf{f} \cdot (\nabla \times \mathbf{A}) = \frac{D}{Dt}\oint_{C(F)} d\mathbf{s} \cdot \mathbf{A}$$

$$\iiint_{V(t)} d\tau\ \nabla\cdot\left(\frac{D_2\mathbf{A}}{Dt}\right) = \oiint_{O(V)} d\mathbf{o}\cdot\left(\frac{D_2\mathbf{A}}{Dt}\right) = \frac{D}{Dt}\oiint_{O(V)} d\mathbf{o}\cdot\mathbf{A} =$$

$$= \iiint_{V(t)} d\tau\ \frac{D_3}{Dt}(\nabla\cdot\mathbf{A}) = \frac{D}{Dt}\iiint_{V(t)} d\tau\ (\nabla\cdot\mathbf{A}) = \frac{D}{Dt}\oiint_{O(V)} d\mathbf{o}\cdot\mathbf{A}$$

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6. Electrodynamics of fluids

We now will introduce the "macroscopic" material derivatives into electromagnetic theory of fluids. Here, it is interesting to derive equations for the vector of electromagnetic momentum, and for electromagnetic energy. We will see that the equation for the vector of electromagnetic momentum corresponds to the equation of motion (3.1) in fluid dynamics. The vorticity equation for electromagnetic momentum corresponds to the hydrodynamic one, and the equation for electromagnetic energy corresponds to the hydrodynamic equation for internal energy (First law of thermodynamics).

We begin with the conventional version of Maxwell´s equations. Originally, they were formulated in a resting frame of reference (Laboratory frame). Denoting the variables in this system by index zero, then these are:

$$\begin{aligned} \frac{\partial \mathbf{B}_0}{\partial t} &= -\nabla \times \mathbf{E}_0, \qquad \nabla \cdot \mathbf{B}_0 = 0 \\ \frac{\partial \mathbf{D}_0}{\partial t} + \mathbf{J}_{e,0} &= \nabla \times \mathbf{H}_0, \qquad \nabla \cdot \mathbf{D}_0 = \rho_e \, . \\ \frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J}_{e,0} &= 0 \end{aligned} \tag{6.1}$$

The notation is as usual: $\mathbf{B}_0$ is the magnetic flux density vector (magnetic induction), $\mathbf{E}_0$ is the electric field vector, $\mathbf{D}_0$ is the displacement vector, $\mathbf{H}_0$ the magnetic field vector, $\mathbf{J}_{e,0}$ is the vector of the electric current density, and ρ_e is charge density.

We introduce the vectors of dielectric polarization $\mathbf{P}_0$ and of magnetization $\mathbf{M}_0$ and use the well-known relations $\mathbf{D}_0 = \varepsilon_0 \mathbf{E}_0 + \mathbf{P}_0$ and $\mathbf{B}_0 = \mu_0 \mathbf{H}_0 + \mathbf{M}_0$. Here ε_0, μ_0 are dielectric constant and magnetic permeability of the vacuum respectively. Normally, parametrizations of $\mathbf{D}_0$ in terms of $\mathbf{E}_0$ and of $\mathbf{H}_0$ in terms of $\mathbf{B}_0$ (closing conditions) are used: $\mathbf{D}_0 = \varepsilon_0 \varepsilon_r \mathbf{E}_0$ and $\mathbf{B}_0 = \mu_0 \mu_r \mathbf{H}_0$. Here ε_r, μ_r are constants (or even tensors) characterizing the electric and magnetic properties of the material under consideration. Putting these equal to one, means that dielectric polarization as well as magnetization are neglected, i. e. the equations are valid in the vacuum. At this point, however, we will avoid this approximation, and we will write down the system of Maxwell´s equations explicitly as:

$$\frac{\partial \mathbf{B}_0}{\partial t} = -\nabla \times \mathbf{E}_0, \qquad \nabla \cdot \mathbf{B}_0 = 0$$
$$\varepsilon_0 \frac{\partial \mathbf{E}_0}{\partial t} + \left(\mathbf{J}_{e,0} + \frac{\partial \mathbf{P}_0}{\partial t} + \frac{1}{\mu_0} \nabla \times \mathbf{M}_0 \right) = \frac{1}{\mu_0} \nabla \times \mathbf{B}_0, \qquad \varepsilon_0 \nabla \cdot \mathbf{E}_0 + \nabla \cdot \mathbf{P}_0 = \rho_e .$$

If we introduce the combined vector of the electric current density $\mathbf{J}_{e,0}$, and the corresponding charge density q_e, defined by

$$\mathbf{J}_{e,0} + \frac{\partial \mathbf{P}_0}{\partial t} + \frac{1}{\mu_0} \nabla \times \mathbf{M}_0 = \mathbf{J}_{e,0}$$
$$\rho_e - \nabla \cdot \mathbf{P}_0 = \varepsilon_0 \nabla \cdot \mathbf{E}_0 = q_e ,$$

Maxwell´s equations then are given as (MAUERSBERGER, 1964, page 23)

$$\frac{\partial \mathbf{B}_0}{\partial t} = -\nabla \times \mathbf{E}_0, \qquad \nabla \cdot \mathbf{B}_0 = 0$$
$$\varepsilon_0 \frac{\partial \mathbf{E}_0}{\partial t} + \mathbf{J}_{e,0} = \frac{1}{\mu_0} \nabla \times \mathbf{B}_0, \qquad \varepsilon_0 \nabla \cdot \mathbf{E}_0 = q_e . \tag{6.1a}$$
$$\frac{\partial q_e}{\partial t} + \nabla \cdot \mathbf{J}_{e,0} = 0$$

We return to the original version (6.1) of Maxwell´s equations. The relativistic transformation of these equations from the resting frame to an arbitrarily moving frame of reference is connected with famous names in physics: Einstein (1905) (single electron), and Minkowski (1908) (general and complete solution of the problem). This transformation corresponds to a transformation from Eulerian coordinates to Lagrangian ones. We will follow Sommerfeld (SOMMERFELD, 1949) and Mauersberger (MAUERSBERGER, 1964, chapter 1.75). In *non-relativistic approximation*, the field vectors are transformed from the resting to the moving system according to

$$\begin{aligned} \mathbf{D}_0 &\approx \mathbf{D}, & \mathbf{E}_0 &\approx \mathbf{E} - \mathbf{V} \times \mathbf{B} \\ \mathbf{B}_0 &\approx \mathbf{B}, & \mathbf{H}_0 &\approx \mathbf{H} + \mathbf{V} \times \mathbf{D} . \\ \mathbf{P}_0 &\approx \mathbf{P}, & \mathbf{M}_0 &\approx \mathbf{M} - \mathbf{V} \times \mathbf{P} \end{aligned} \tag{6.2}$$

If we introduce the first two members of (6.2) in (6.1), equations (6.1) transform to

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{V} \times \mathbf{B}) = -\nabla \times \mathbf{E}, \qquad \nabla \cdot \mathbf{B} = 0$$
$$\frac{\partial \mathbf{D}}{\partial t} - \nabla \times (\mathbf{V} \times \mathbf{D}) + \mathbf{J}_{e,0} = \nabla \times \mathbf{H}, \qquad \nabla \cdot \mathbf{D} = \rho_e .$$

Now, we introduce the following vector relationship, valid for any vector $\mathbf{A}$,

$$-\nabla \times (\mathbf{V} \times \mathbf{A}) = \mathbf{V} \cdot \nabla \mathbf{A} + (\nabla \cdot \mathbf{V})\mathbf{A} - \mathbf{A} \cdot (\nabla \mathbf{V}) - (\nabla \cdot \mathbf{A})\mathbf{V} . \tag{6.3}$$

Then, the “Lagrange” version of Maxwell’s equations (6.1), by applying (1.2), is given by

$$\frac{D_2\mathbf{B}}{Dt} = -\nabla \times \mathbf{E}, \qquad \nabla \cdot \mathbf{B} = 0,$$

$$\frac{D_2\mathbf{D}}{Dt} - (\nabla \cdot \mathbf{D})\mathbf{V} + \mathbf{J}_{e,0} = \nabla \times \mathbf{H}, \qquad \nabla \cdot \mathbf{D} = \rho_e .$$

In the moving system, we have

$$\mathbf{J}_e = \mathbf{J}_{e,0} - \rho_e \mathbf{V} . \tag{6.4}$$

The final set of equations, applicable to fluid dynamics, is given by

$$\begin{aligned} \frac{D_2\mathbf{B}}{Dt} &= -\nabla \times \mathbf{E}, \qquad \nabla \cdot \mathbf{B} = 0 \\ \frac{D_2\mathbf{D}}{Dt} + \mathbf{J}_e &= \nabla \times \mathbf{H}, \qquad \nabla \cdot \mathbf{D} = \rho_e \end{aligned} \tag{6.5}$$

Applying the $\nabla \cdot$ - operation on the last member of (6.5), and observing that

$$\nabla \cdot \frac{D_2\mathbf{D}}{Dt} = \frac{D_3\nabla \cdot \mathbf{D}}{Dt} = \frac{D_3\rho_e}{Dt},$$

the equation of continuity is given by

$$\frac{D_3\rho_e}{Dt} + \nabla \cdot \mathbf{J}_e = 0 . \tag{6.6}$$

If there is only a translation of coordinates with a constant velocity vector $\mathbf{V}$, the operator D_2/Dt in (6.5) is replaced by D/Dt. That case played an important role in the early theory of special relativity.

We mention that equations (6.5), (6.6) can be transformed to Lagrangian coordinates by applying (A1.7a), (A1.8) of Appendix 1 together with the definition of a covariant derivative. Then the analogy to the original equations in a laboratory frame becomes closer. Additionally we learn that $\mathbf{B}, \mathbf{D}$, and $\mathbf{J}_e$ are contravariant vectors, while $\mathbf{E}, \mathbf{H}$ are covariant ones. This can be seen also in equations (6.7).

Integrating over material surface areas and over material volumes respectively the integral versions of (6.5) and (6.6) are

$$\begin{aligned} &\frac{D}{Dt}\left[\iint_{F(t)} d\mathbf{f} \cdot \mathbf{B}\right] = -\oint_{C(t)} d\mathbf{s} \cdot \mathbf{E}, \qquad \oiint_{O(t)} d\mathbf{o} \cdot \mathbf{B} = 0 \\ &\frac{D}{Dt}\left[\iint_{F(t)} d\mathbf{f} \cdot \mathbf{D}\right] + \iint_{F(t)} d\mathbf{f} \cdot \mathbf{J}_e = \oint_{C(t)} d\mathbf{s} \cdot \mathbf{H}, \qquad \oiint_{O(t)} d\mathbf{o} \cdot \mathbf{D} = \iiint_{V(t)} d\tau\, \rho_e \\ &\frac{D}{Dt}\left[\iiint_{V(t)} d\tau\, \rho_e\right] = -\oiint_{O(t)} d\mathbf{o} \cdot \mathbf{J}_e \end{aligned} \tag{6.7}$$

Here, we have Faraday´s , Ampere´s, and Coulomb´s laws together with the law for absence of magnetic poles, and the equation of continuity for electric charge density. If Ampere and Faraday had been able to perform their experiments while walking, they would have drawn their conclusions equivalently to equations (6.7).

In atmospheric electrodynamics, we will apply the equations that are valid in the vacuum (no dielectric polarization, no magnetization of the fluid) and write (6.5) as

$$\frac{D_2\mathbf{B}}{Dt} = -\nabla\times\mathbf{E}, \qquad \nabla\cdot\mathbf{B} = 0$$
$$\frac{D_2(\varepsilon_0\mathbf{E})}{Dt} + \mathbf{J}_e = \nabla\times\frac{\mathbf{B}}{\mu_0}, \qquad \nabla\cdot(\varepsilon_0\mathbf{E}) = \rho_e \quad . \tag{6.8}$$

With regard to the equations to be derived, we list special properties of the electromagnetic field:

$$\mathbf{p} = \varepsilon_0\mathbf{E}\times\mathbf{B}\ \left[(kg/m^3)(m/s)\right],\ \ \mathbf{S} = \mathbf{E}\times\frac{\mathbf{B}}{\mu_0}\ \left[J/m^2s\right],\ \ \mathbf{K}_\mathrm{L} = \rho_e\mathbf{E} + \mathbf{J}_e\times\mathbf{B}\ \left[J/m^4\right],$$
$$\mathbf{T} = \varepsilon_0\mathbf{EE} + \frac{1}{\mu_0}\mathbf{BB} - \eta\mathbf{I}\ \left[J/m^3\right],\ \ \eta = \frac{1}{2}\left(\varepsilon_0\mathbf{E}^2 + \frac{\mathbf{B}^2}{\mu_0}\right)\ \left[J/m^3\right],\ \ \varepsilon_0\mu_0c_0^2 = 1 \quad . \tag{6.9}$$

Here, **p** is the vector of momentum, **S** is Poynting´s vector, $\mathbf{K}_L$ is Lorentz´ force, **T** Maxwell´s stress tensor, and η is energy density. Moreover, ε_0 is dielectric constant, and μ_0 is magnetic permeability.

The equation for electromagnetic momentum density **p** is obtained from:

$$\frac{D_2}{Dt}(\varepsilon_0\mathbf{E})\times\mathbf{B} - \frac{D_2\mathbf{B}}{Dt}\times(\varepsilon_0\mathbf{E}) = \left(\nabla\times\frac{\mathbf{B}}{\mu_0}\right)\times\mathbf{B} - \mathbf{J}_e\times\mathbf{B} + (\nabla\times\mathbf{E})\times(\varepsilon_0\mathbf{E})$$

i.e.

$$\frac{D\mathbf{p}}{Dt} + (\nabla\cdot\mathbf{V})\mathbf{p} + (\nabla\mathbf{V})\cdot\mathbf{p} = -\mathbf{K}_\mathrm{L} + \nabla\cdot\mathbf{T}. \tag{6.10}$$

The result can be obtained without difficulty, though we leave the derivation of (6.10) as well as of (6.13) to the reader. As a hint we note that for any tensor $\mathbf{\Phi}$, and so for $\nabla\mathbf{V}$ with the first scalar $\Phi_1 = \nabla\cdot\mathbf{V}$,

$$\mathbf{a}\cdot\mathbf{\Phi}\times\mathbf{b} - \mathbf{b}\cdot\mathbf{\Phi}\times\mathbf{a} = \Phi_1(\mathbf{a}\times\mathbf{b}) - \mathbf{\Phi}\cdot(\mathbf{a}\times\mathbf{b}).$$

It is more convenient to introduce the vector $[\upsilon\mathbf{p}] = m/s$. Using in (6.6) the equation of continuity of the fluid, $\nabla\cdot\mathbf{V} = \rho D\upsilon/Dt$, we find that the following equation (6.11) is equivalent to the hydrodynamic equation of motion (3.1):

$$\frac{D_1\upsilon\mathbf{p}}{Dt} = -\upsilon\mathbf{K}_\mathrm{L} + \upsilon\nabla\cdot\mathbf{T}, \tag{6.11}$$

$$\frac{D_1\mathbf{V}_\mathbf{a}}{Dt} = \nabla\left(\mathbf{V}_\mathbf{a}^2/2 - \phi^{(A)}\right) - \upsilon\nabla p + \upsilon\nabla\cdot\mathbf{F}. \tag{3.1}$$

It is known that Maxwell´s stress tensor **T** in electromagnetic theory act in the same way as the friction tensor **F** does in fluid dynamics. If we add both equations, we get a momentum equation for a fluid in which mechanical and electromagnetic momentum act together.

The vorticity-equation for electromagnetic momentum is equivalent to the vorticity equation of fluid dynamics:

$$\frac{D_2}{Dt}(\nabla\times\upsilon\mathbf{p}) = \nabla\times\left(-\upsilon\mathbf{K}_\mathrm{L} + \upsilon\nabla\cdot\mathbf{T}\right) \tag{6.12}$$

$$\frac{D_2}{Dt}(\nabla\times\mathbf{V}_\mathbf{a}) = \nabla\times\left(-\upsilon\nabla p + \upsilon\nabla\cdot\mathbf{F}\right) \tag{3.2}$$

We conclude, that all results obtained in **4.**, concerning vorticity theorems, apply equally well to the field of electromagnetic momentum.

The equation for electromagnetic energy density η is obtained from:

$$\frac{D_2}{Dt}(\varepsilon_0 \mathbf{E}) \cdot \mathbf{E} + \frac{D_2}{Dt}\left(\frac{\mathbf{B}}{\mu_0}\right) \cdot \mathbf{B} = \left(\nabla \times \frac{\mathbf{B}}{\mu_0}\right) \cdot \mathbf{E} - \mathbf{J}_e \cdot \mathbf{E} - (\nabla \times \mathbf{E}) \cdot \frac{\mathbf{B}}{\mu_0}$$

i.e.

$$\frac{D_3 \eta}{Dt} + \nabla \cdot \mathbf{S} = -\mathbf{J}_e \cdot \mathbf{E} + \mathbf{T} \cdot\cdot \nabla \mathbf{V} \qquad (6.13)$$

We compare this result with the first law of thermodynamics for a fluid

$$\rho \frac{Du}{Dt} + \nabla \cdot \mathbf{J}_q = \frac{D_3 \rho u}{Dt} + \nabla \cdot \mathbf{J}_q = -p \nabla \cdot \mathbf{V} + \mathbf{F} \cdot\cdot \nabla \mathbf{V} . \qquad (6.14)$$

Here, u is internal energy, $\mathbf{J}_q$ is the vector of heat flux, and $\mathbf{F} \cdot\cdot \nabla \mathbf{V} = \rho\delta$ is energy dissipation due to friction. Both equations have the same mathematical "structure". Again, if we add both equations, we get an energy equation for a fluid in which mechanical and electromagnetic momentum act together:

$$\frac{D_3}{Dt}(\rho u + \eta) + \nabla \cdot (\mathbf{J}_q + \mathbf{S}) = -p \nabla \cdot \mathbf{V} - \mathbf{J}_e \cdot \mathbf{E} + (\mathbf{F} + \mathbf{T}) \cdot\cdot \nabla \mathbf{V} . \qquad (6.15)$$

Additionally, we can find an electromagnetic entropy equation, similar to that in thermodynamics of fluids

$$\frac{D_3 \rho s}{Dt} + \nabla \cdot \left(\frac{\mathbf{J_q}}{T}\right) = \frac{1}{T}\left(-\frac{\mathbf{J_q}}{T} \cdot \nabla T + \mathbf{F} \cdot\cdot \nabla \mathbf{V}\right) = \rho\sigma(s) . \qquad (3.14)$$

If we define an integrating denominator T^* (a temperature, different from T) by the equation

$$\frac{1}{T^*} \frac{D_3 \eta}{Dt} = \frac{D_3 \chi}{Dt} , \qquad (6.16)$$

where χ is entropy,

$$\frac{D_3 \chi}{Dt} + \nabla \cdot \left(\frac{\mathbf{S}}{T^*}\right) = \frac{1}{T^*}\left(-\frac{\mathbf{S}}{T^*} \cdot \nabla T^* + \mathbf{T} \cdot\cdot \nabla \mathbf{V}\right) = \rho\sigma(\chi) \qquad (6.17)$$

Addition of (3.14) and (3.17), we find for the combined system the entropy equation as

$$\frac{D_3}{Dt}(\rho s + \chi) + \nabla \cdot \left(\frac{\mathbf{J_q}}{T} + \frac{\mathbf{S}}{T^*}\right) = \frac{1}{T}\left(-\frac{\mathbf{J_q}}{T} \cdot \nabla T + \mathbf{F} \cdot\cdot \nabla \mathbf{V}\right) + \frac{1}{T^*}\left(-\frac{\mathbf{S}}{T^*} \cdot \nabla T^* + \mathbf{T} \cdot\cdot \nabla \mathbf{V}\right) \qquad (6.18)$$

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